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Orbifold construction of the modes of the Poincaré dodecahedral space

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Abstract

We provide a new construction of the modes of the Poincaré dodecahedral space S^3/I^* . The construction uses the Hopf map, Maxwell's multipole vectors and orbifolds. In particular, the $*235$ -orbifold serves as a parameter space for the modes of S^3/I^* , shedding new light on the geometrical significance of the dimension of each space of k -modes, as well as on the modes themselves.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Cosmic topology [4] considers big bang models where space is multiconnected, rather than simply connected like in their standard counterparts. Such multiconnected models (MCMs) have been compared to the large scale structures like the distributions of galaxies or clusters but such tests have apparently reached their limits given the present status of the large scale surveys. On the other hand, many recent works have examined the possible signature of MCMs in the characteristics of the cosmic microwave background (CMB). For example, the Poincaré dodecahedral space (PDS) provided a concrete model that naturally explained the weak broad-scale CMB fluctuations ([9, 20]; see also [23, 24]) while maintaining good agreement with the curvature estimates favored at the time [18]. Other models have also been invoked [1].

Estimating the characteristics of CMB fluctuations, in a given model, requires the knowledge of the eigenmodes of the Laplacian on the spatial sections. In the usual (simply connected) big bang models, such sections are copies of $M = S^3, \mathbb{R}^3$ or H^3 , according to the spatial curvature, and their eigenmodes are well known analytically. In MCMs, the spatial sections are multiconnected spaces.

In the positive curvature case, the most cosmologically relevant examples take the form S^3/Γ^* , the quotients of S^3 by a binary polyhedral group Γ^* . Their eigenmodes are not known in general. Thus, the first estimates of the statistics of the CMB fluctuations in MCMs

[20] used numerical estimates. Those can be performed only for the first eigenmodes, thus limiting the validity of the possible comparisons with observations. This motivated the search for analytical expressions of the eigenmodes of the spherical spaces. Such modes may be seen as the Γ^* -invariant solutions of the Helmholtz equation in the universal cover S^3 . Their numeration and degeneracy were given by Ikeda [5]. Recent works [6–8, 10, 11] have provided various means to calculate them, which were used in [9] to compare the CMB data with the MCM predictions.

Since these constructions of the eigenmodes of S^3/Γ^* are based on group theoretical arguments, they naturally involve the Wigner D-functions $T_{k;m_1,m_2}$. As a consequence, the result of the calculations, i.e., the desired eigenmodes, is given by their expansion with respect to the Wigner functions, which provide a basis for the eigenmodes of S^3 , rather than with respect to the more usual hyperspherical harmonics $\mathcal{Y}_{k\ell m}$. This has no fundamental importance, but for historical and for practical reasons the CMB fluctuations are expanded in the usual spherical harmonics $Y_{\ell m}$ of the celestial sphere S^2 . It is rather natural and easy to calculate the coefficients of the CMB fluctuations in their $Y_{\ell m}$ expansion from the spatial modes expressed in their $\mathcal{Y}_{k\ell m}$ expansion. On the other hand, the same calculation from the spatial modes in their $T_{k;m_1,m_2}$ expansion is much more tedious: it requires a conversion from the $T_{k;m_1,m_2}$ basis to the $\mathcal{Y}_{k\ell m}$ basis. This simple change of basis presents no fundamental difficulty, but involves the calculation of families of Clebsch–Gordan coefficients which requires large amounts of computer time and memory. This put limits on the practical validity of the calculations. Thus, although the problem is formally solved, practical considerations motivate the search for alternative methods.

It has been recognized in recent years [13, 14, 16, 17, 19, 21, 25] that the eigenmodes of S^2 , and their statistics, can be expressed as multipole harmonics rather than through their $Y_{\ell m}$ expansion. Various algorithms have already been developed for analyzing the CMB data in this formalism, and it has been recognized that it may offer some advantages to express deviations from isotropy, precisely the kind of effects expected in MCMs (see, e.g., [2]). Thus it seems very promising to analyze the CMB fluctuations predicted from MCMs in the multipole formalism rather than with the familiar $Y_{\ell m}$ expansion. A first analysis has been accomplished recently by [3], who point out however the absence of predictions from the MCMs, i.e., an expression of the predicted CMB fluctuations in the multipole formalism. The work presented here offers a first step in this direction: instead of calculating the eigenmodes of a spherical space (including S^3 itself) in their $\mathcal{Y}_{k\ell m}$ or $T_{k;m_1,m_2}$ expression, we construct them directly from some selected S^2 multipole eigenmodes. More precisely, we show that the eigenmodes of S^3/Γ^* are ‘lifts’ of the Γ -invariant multipole eigenmodes of S^2 .

Our construction involves a new point of view. It uses the Hopf map $S^3 \rightarrow S^2$, the multipole eigenmodes, and orbifolds. Providing a direct link between the spatial eigenmodes and the multipole vectors of S^2 , it opens the possibility of obtaining a direct estimate of the CMB fluctuations as multipole harmonics, from our expression of the spatial eigenmodes in a MCM, without requiring calculation-intensive intermediate steps. Beside the possibility of extending the validity of the previous tests (which becomes necessary with the increased precision of the CMB data), this could permit the implementation of tests of a different nature, because of the distinct discriminative power of the multipole analysis compared to the familiar $Y_{\ell m}$ expansion of the CMB fluctuations (see [2, 15, 22]). In particular, the already existing analyses of the CMB data in the multipole formalism could be used as new constraints (or confirmations) for the MCMs; in particular, regarding the characteristic anisotropies predicted by the MCMs, a task presently out of reach with the usual spherical harmonics formalism.

Section 2 reviews the Hopf map and uses it to lift eigenmodes from S^2 to S^3 . Section 3 uses twist operators to extend the lifted modes to a full eigenbasis for S^3 . Section 4 generalizes

the preceding results from the modes of S^3 to the modes of a spherical space S^3/Γ^* , showing that the latter all come from the lifts of those eigenmodes of S^2 that are invariant under the corresponding (non-binary) polyhedral group Γ . We then turn to a detailed study of the Γ -invariant modes of S^2 . Section 5 recalls Maxwell’s multipole vector approach and uses it to associate each mode of S^2/Γ with a Γ -invariant set of multipole directions. Restricting attention to the case that Γ is the icosahedral group, section 6 introduces the concept of an orbifold and re-interprets a Γ -invariant set of multipole directions as a (much smaller) set of points in the *235-orbifold, which serves as the parameter space. Section 7 pulls together the results of the preceding sections to summarize the construction of the modes of the PDS and state the dimension of the mode space for each k . This construction provides a new demonstration of Ikeda’s formula [5] and sheds additional light on its geometrical significance.

2. From S^2 to S^3 : lifting with the Hopf map

2.1. Spheres

We parameterize the circle S^1 as the set of points $\alpha \in \mathbb{C}$ of unit norm $\alpha\bar{\alpha} = 1$. The relationship between the complex coordinate α and the usual Cartesian coordinates (x, y) is the natural one: $\alpha = x + iy$.

We parameterize the 2-sphere S^2 as the set of points $(x, y, z) \in \mathbb{R}^3$ of unit norm $x^2 + y^2 + z^2 = 1$.

We parameterize the 3-sphere S^3 as the unit sphere in \mathbb{C}^2 : the set of points $(\alpha, \beta) \in \mathbb{C}^2$ of unit norm $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. Hereafter, we will always assume that this normalization relation holds. The relationship between the complex coordinates (α, β) and the usual Cartesian coordinates (x, y, z, w) is the natural one: $\alpha = x + iy$ and $\beta = z + iw$.

2.2. The Hopf fibration

In S^3 , simultaneous rotation in the α - and β -planes defines the *Hopf flow* $H_t : S^3 \rightarrow S^3$,

$$H_t(\alpha, \beta) \equiv (e^{it}\alpha, e^{it}\beta). \tag{1}$$

The Hopf flow is homogeneous in the sense that it looks the same at all points. An orbit

$$\{(e^{it}\alpha, e^{it}\beta) \mid 0 \leq t < 2\pi\} \tag{2}$$

is a great circle on S^3 called a *Clifford parallel* (figure 1). Collectively, the Clifford parallels comprise the *Hopf fibration* of S^3 . The fibers carry Clifford’s name because William Kingdon Clifford (1845–1879) discovered them before Heinz Hopf (1894–1971) was born. However, while Clifford understood the fibration quite well, he did not, as far as we know, go on to consider the quotient map (equation (3)).

As we walk along any given Clifford parallel $(e^{it}\alpha, e^{it}\beta)$, the ratio of its coordinates $\frac{e^{it}\alpha}{e^{it}\beta}$ remains a constant $\frac{\alpha}{\beta}$, independent of t . The ratio $\frac{\alpha}{\beta}$ labels uniquely each Clifford parallel, taking values in the extended complex numbers $\mathbb{C} \cup \{\infty\}$, where ∞ represents the ratio $\frac{\alpha}{\beta} = \frac{1}{0}$. The extended complex numbers may be visualized as a Riemann sphere, proving that the Clifford parallels are in one-to-one correspondence with the points of a topological 2-sphere S .

The *Hopf map* is defined as sending any point (α, β) of S^3 to the fiber its belong to, i.e., the point of S labeled by $\frac{\alpha}{\beta}$. Composing with a natural map from S to the unit 2-sphere S^2 gives an explicit formula for the Hopf map,

$$p : S^3 \rightarrow S^2(\alpha, \beta) \rightarrow p(\alpha, \beta) = (x, y, z) = (\alpha\bar{\beta} + \bar{\alpha}\beta, -i(\alpha\bar{\beta} - \bar{\alpha}\beta), \beta\bar{\beta} - \alpha\bar{\alpha}). \tag{3}$$

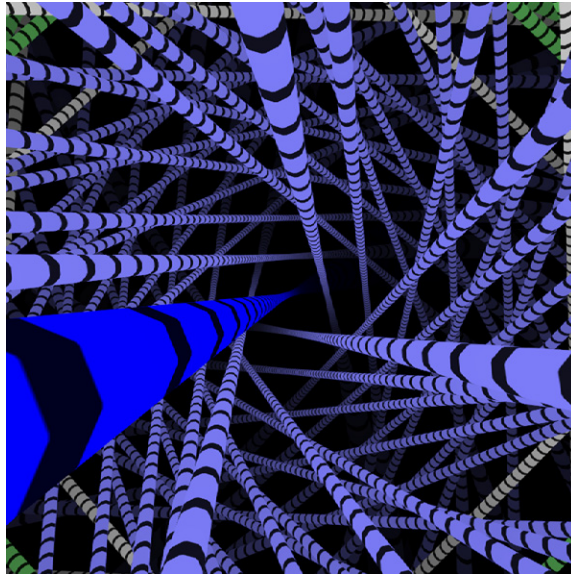


Figure 1. A computer generated view of the Clifford parallels in S^3 .

It is easy to check that $x^2 + y^2 + z^2 = 1$, confirming that the Hopf map p sends S^3 to the unit 2-sphere.

2.3. Lifts of functions

Any given function f on S^2 lifts to a function F on S^3 by composition with the Hopf map p from equation (3),

$$F : S^3 \xrightarrow{p} S^2 \xrightarrow{f} \mathbb{R}. \tag{4}$$

In other words, $F = p^*f$ is the pull-back of f by p : explicitly,

$$F(\alpha, \beta) \equiv f(p(\alpha, \beta)) = f(\alpha\bar{\beta} + \bar{\alpha}\beta, -i(\alpha\bar{\beta} - \bar{\alpha}\beta), \beta\bar{\beta} - \alpha\bar{\alpha}). \tag{5}$$

For example, the quadratic polynomial

$$f(x, y, z) = x^2 - y^2 \tag{6}$$

lifts to the quartic polynomial

$$F(\alpha, \beta) = (\alpha\bar{\beta} + \bar{\alpha}\beta)^2 - (-i(\alpha\bar{\beta} - \bar{\alpha}\beta))^2 = 2(\alpha^2\bar{\beta}^2 + \bar{\alpha}^2\beta^2). \tag{7}$$

Definition 2.3.1. We call a function $F : S^3 \rightarrow \mathbb{R}$ vertical if it is constant along every Clifford parallel (formula (2)).

For every function $f : S^2 \rightarrow \mathbb{R}$, the construction of the lift $F(\alpha, \beta) = f(p(\alpha, \beta))$ guarantees that F is vertical.

Proposition 2.3.2. The Hopf map lifts a polynomial $f : S^2 \rightarrow \mathbb{R}$ of degree ℓ to a polynomial $F : S^3 \rightarrow \mathbb{R}$ of degree 2ℓ .

Proof. The lifting formula (5) doubles the degree of any polynomial. □

3. Eigenmodes

3.1. Basic definitions

Definition 3.1.1. An ℓ -eigenmode is an eigenmode $f : S^2 \rightarrow \mathbb{R}$ of the Laplacian, with eigenvalue $\lambda_\ell = \ell(\ell + 1)$.

An ℓ -eigenmode is a solution of the Helmholtz equation

$$\Delta_{S^2} f = \ell(\ell + 1) f. \tag{8}$$

The index ℓ takes values in the set $\{0, 1, 2, \dots\}$. For each ℓ , the ℓ -eigenmodes (which are the usual spherical harmonics) form a vector space V^ℓ of dimension $2\ell + 1$.

Definition 3.1.2. A k -eigenmode is an eigenmode $F : S^3 \rightarrow \mathbb{R}$ of the Laplacian with eigenvalue $\lambda_k = k(k + 2)$.

A k -eigenmode is a solution of the Helmholtz equation

$$\Delta_{S^3} F = k(k + 2) F. \tag{9}$$

The index k takes values in the set $\{0, 1, 2, \dots\}$. For each k , the k -eigenmodes form a vector space V^k of dimension $(k + 1)^2$.

3.2. Eigenmodes of S^2 define eigenmodes of S^3

Proposition 3.2.1. An ℓ -eigenmode f on the unit 2-sphere lifts to a k -eigenmode F on the unit 3-sphere, with $k = 2\ell$.

Proof. It is well known that the ℓ -eigenmodes are precisely the homogenous harmonic polynomials of degree ℓ on \mathbb{R}^3 , with domain restricted to the unit 2-sphere. Similarly the k -eigenmodes are the homogeneous harmonic polynomials of degree k on \mathbb{R}^4 , with domain restricted to the unit 3-sphere. A harmonic function on \mathbb{R}^4 satisfies

$$\Delta_{R^4} F \equiv 4(\partial_\alpha \partial_{\bar{\alpha}} + \partial_\beta \partial_{\bar{\beta}}) F = 0. \tag{10}$$

When F is the pull-back of f given by (5), direct calculations give

$$\Delta_{R^4} F(\alpha, \beta) = (\partial_x \partial_x + \partial_y \partial_y + \partial_z \partial_z) f(x, y, z) = \Delta_{R^3} f(x, y, z). \tag{11}$$

Thus, the pull-back of a harmonic function on \mathbb{R}^3 is a harmonic function on \mathbb{R}^4 , and therefore the pull-back of an eigenmode of Δ_{S^2} is an eigenmode of Δ_{S^3} . Together with proposition 2.3.2, this completes the proof. \square

Notation 3.2.2. Let $Y_{\ell m}$ denote the usual spherical harmonics on S^2 . For example, the $Y_{2,m}$ may be expressed as harmonic polynomials as follows:

| | Trigonometric | Polynomial |
|------------|--|---|
| $Y_{2,+2}$ | $\sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2mi\varphi}$ | $\sqrt{\frac{15}{32\pi}} (x + iy)^2$ |
| $Y_{2,+1}$ | $\sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{i\varphi}$ | $\sqrt{\frac{15}{8\pi}} z(x + iy)$ |
| $Y_{2,0}$ | $\sqrt{\frac{5}{16\pi}} (1 - 3 \cos^2 \theta)$ | $\sqrt{\frac{5}{16\pi}} (x^2 + y^2 - 2z^2)$ |
| $Y_{2,-1}$ | $\sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{-i\varphi}$ | $\sqrt{\frac{15}{8\pi}} z(x - iy)$ |
| $Y_{2,-2}$ | $\sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{-2i\varphi}$ | $\sqrt{\frac{15}{32\pi}} (x - iy)^2$ |

Let $\mathcal{Y}_{km0} = Y_{\ell m} \circ p$, with $k = 2\ell$, denote the pullback of $Y_{\ell m}$ under the action of the Hopf map (3). In accordance with proposition 2.3.2, its degree is k . For example, $Y_{2,0} = \sqrt{\frac{5}{16\pi}}(x^2 + y^2 - 2z^2)$ lifts to $\mathcal{Y}_{4,0,0} = \sqrt{\frac{5}{4\pi}}(\alpha^2\bar{\alpha}^2 - 4\alpha\bar{\alpha}\beta\bar{\beta} + \beta^2\bar{\beta}^2)$ of degree 4.

The \mathcal{Y}_{km0} are simply the realization of the $Y_{\ell m}$ on the abstract 2-sphere S of Clifford parallels. As such, the linear independence of the $Y_{\ell m}$ immediately implies the linear independence of the \mathcal{Y}_{km0} as well.

3.3. Twist

Each \mathcal{Y}_{km0} is constant along Clifford parallels, but more general functions are not. As we take one trip around a Clifford parallel $(e^{it}\alpha_0, e^{it}\beta_0)$, $0 \leq t \leq 2\pi$, the value of the monomial $\alpha^a\bar{\alpha}^b\beta^c\bar{\beta}^d$ varies as $e^{i(a-b+c-d)t}$ times the constant $\alpha_0^a\bar{\alpha}_0^b\beta_0^c\bar{\beta}_0^d$. In other words, the value of a typical monomial $\alpha^a\bar{\alpha}^b\beta^c\bar{\beta}^d$ rotates counterclockwise $(a - b + c - d)$ times in the complex plane as we take one trip around any Clifford parallel. The graph of the monomial is a helix sitting over the Clifford parallel, motivating the following definition.

Definition 3.3.1. *The twist of a monomial $\alpha^a\bar{\alpha}^b\beta^c\bar{\beta}^d$ is the power of the unbarred variables minus the power of the barred variables, i.e. $a - b + c - d$. The twist of a polynomial is the common twist of its terms, in cases where those twists all agree; otherwise it is undefined.*

Proposition 3.3.2. *The polynomials of well-defined twist (including all monomials) are precisely the eigenmodes of the operator*

$$\alpha\partial_\alpha - \bar{\alpha}\partial_{\bar{\alpha}} + \beta\partial_\beta - \bar{\beta}\partial_{\bar{\beta}}, \tag{12}$$

with the twist as eigenvalue.

Proof. Apply the operator to $\alpha^a\bar{\alpha}^b\beta^c\bar{\beta}^d$ and observe the result. □

Geometrically, operator (12) is essentially the directional derivative operator in the direction of the Clifford parallels, the only difference being that the directional derivative includes a factor of i that operator (12) does not, because the complex-valued derivative is 90° out of phase with the value of the function itself.

Because we consider modes of even k only, the twist will always be even. Henceforth, for notational convenience, we shall take our twist-measuring operator to be

$$Z = \frac{1}{2}(\alpha\partial_\alpha - \bar{\alpha}\partial_{\bar{\alpha}} + \beta\partial_\beta - \bar{\beta}\partial_{\bar{\beta}}). \tag{13}$$

The ad hoc factor of $1/2$ transforms the range of eigenmodes from even integers to all integers.

3.4. Siblings and the twist operators

The twist operators

$$\text{twist} \equiv -\beta\partial_{\bar{\alpha}} + \alpha\partial_{\bar{\beta}} \quad \overline{\text{twist}} \equiv -\bar{\beta}\partial_\alpha + \bar{\alpha}\partial_\beta \tag{14}$$

(defined in [12]) increase and decrease a function's twist. That is, the twist operator converts an n -eigenmode of Z to an $(n + 1)$ -eigenmode of Z , and inversely for $\overline{\text{twist}}$. Here is the proof: it is easy to check that the commutator $[Z, \text{twist}] = \text{twist}$, so given $ZF = \lambda F$ it follows that

$$Z(\text{twist } F) = (Z\text{twist})F = (\text{twist } Z + \text{twist})F = (\lambda + 1)(\text{twist } F).$$

Thus the operator twist increases by one unit the eigenvalue of an eigenfunction of Z , and similarly $\overline{\text{twist}}$ decreases it by one unit.

Because Δ_{S^3} and twist commute (see [12]), the twist operator transforms each k -eigenmode into another k -eigenmode.

Being vertical, each \mathcal{Y}_{km0} is an eigenmode of Z with eigenvalue 0. Repeatedly applying the operator twist gives eigenmodes of Z with eigenvalues $1, 2, \dots, k/2$ (k is even), while repeatedly applying the operator $\overline{\text{twist}}$ gives modes with eigenvalues $-1, -2, \dots, -k/2$. Why do the sequences stop at $n = \pm k/2$? The explanation is as follows. When written as a polynomial in the complex variables $\{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$, the original vertical mode \mathcal{Y}_{km0} contains equal powers of the barred variables $\bar{\alpha}$ and $\bar{\beta}$ and the unbarred variables α and β . The operator twist replaces a barred variable with an unbarred one, keeping the degree constant while increasing the difference $\#unbarred - \#barred$ by two. After $\frac{k}{2}$ applications of the twist operator, the polynomial $\text{twist}^{k/2}\mathcal{Y}_{km0}$ contains unbarred variables alone: it has maximal positive twist and further application of the twist operator collapses it to zero. Analogously, the $\overline{\text{twist}}$ operator converts unbarred variables to barred ones, until $\overline{\text{twist}}^{k/2}\mathcal{Y}_{km0}$ consists of barred variables alone, after which further applications of $\overline{\text{twist}}$ collapse it to zero.

Let \mathcal{Y}_{kmn} be the resulting modes. That is, for $n = 1, 2, \dots, k/2$, define

$$\mathcal{Y}_{k,m,+n} = \text{twist}^n \mathcal{Y}_{km0} \quad \mathcal{Y}_{k,m,-n} = \overline{\text{twist}}^n \mathcal{Y}_{km0}. \tag{15}$$

Each $\mathcal{Y}_{k,m,n}$ is simultaneously a k -eigenmode of the Laplacian and an n -eigenmode of Z .

The modes $\{\mathcal{Y}_{k,m,n}\}_{n=-k/2 \dots k/2}$, being eigenmodes with different eigenvalues, are linearly independent [12]. Conclusion: each $Y_{\ell m}$ generates, via the lift from S^2 to S^3 (sections 2.3 and 3.2) and the twist operators, a $(k + 1)$ -dimensional vector space \mathcal{V}^{km} of k -modes, with basis $\{\mathcal{Y}_{k,m,n}\}_{n=-k/2 \dots k/2}$ (see table 1). Thus the $2\ell + 1 = k + 1$ spherical harmonics $Y_{\ell m}$ generate the complete vector space of k -eigenmodes of S^3 ,

$$\mathcal{V}^k = \bigoplus_m \mathcal{V}^{km},$$

with basis $\{\mathcal{Y}_{k,m,n}\}_{m=-k/2 \dots k/2, n=-k/2 \dots k/2}$, and thus of dimension $(k + 1)^2$.

Proposition 3.4.1. $\mathcal{Y}_{k,m,n} = \overline{\mathcal{Y}_{k,-m,-n}}$.

Proof. Each $\mathcal{Y}_{k,+m,0}$ is conjugate to the corresponding $\mathcal{Y}_{k,-m,0}$ because they are lifts of the standard two-dimensional spherical harmonics $Y_{\ell,+m}$ and $Y_{\ell,-m}$ which have this symmetry. The twist operators (14) are complex conjugates of one another by construction. Therefore when $n \geq 0$,

$$\overline{\mathcal{Y}_{k,m,n}} = \overline{\text{twist}^n \mathcal{Y}_{k,m,0}} = \overline{\text{twist}^n} \mathcal{Y}_{k,-m,0} = \mathcal{Y}_{k,-m,-n}, \tag{16}$$

and similarly when $n \leq 0$. □

Proposition 3.4.2. By choosing complex-conjugate coefficients $c_{k,m,n} = \overline{c_{k,-m,-n}}$ one may recover the real-valued modes of S^3 as

$$c_{kmn}\mathcal{Y}_{k,m,n} + \overline{c_{kmn}}\mathcal{Y}_{k,-m,-n}. \tag{17}$$

In particular, whenever m and n are not both zero, the modes

$$\mathcal{Y}_{k,m,n} + \mathcal{Y}_{k,-m,-n} \tag{18}$$

$$i\mathcal{Y}_{k,m,n} - i\mathcal{Y}_{k,-m,-n} \tag{19}$$

are independent real-valued modes, analogous to cosine and sine, respectively.

Proof. The mode (17) is its own complex conjugate,

$$\overline{c_{kmn}\mathcal{Y}_{k,m,n} + \overline{c_{kmn}}\mathcal{Y}_{k,-m,-n}} = c_{kmn}\mathcal{Y}_{k,m,n} + \overline{c_{kmn}}\mathcal{Y}_{k,-m,-n} \tag{20}$$

and therefore real. □

Table 1. Each ℓ -eigenmode $Y_{\ell,m}$ of S^2 (i.e. each spherical harmonic; middle column, lower entries) lifts via the Hopf map (\uparrow) to a 0-twist k -eigenmode $\mathcal{Y}_{k,m,0}$ of S^3 (middle column, upper entries), with $k = 2\ell$. The positive twist operator (\rightarrow) then takes $\mathcal{Y}_{k,m,0}$ to its $\frac{k}{2}$ positively twisted siblings (right side) while the negative twist operator (\leftarrow) takes $\mathcal{Y}_{k,m,0}$ to its $\frac{k}{2}$ negatively twisted siblings (left side), for a total of $(k + 1)^2$ linearly independent modes.

| | | | | | | | | | | | | |
|------------------------------|--------------|----------|--------------|----------------------------|--------------|---------------------------|---------------|----------------------------|---------------|----------|---------------|------------------------------|
| $\mathcal{Y}_{k,+\ell,-k/2}$ | \leftarrow | \cdots | \leftarrow | $\mathcal{Y}_{k,+\ell,-1}$ | \leftarrow | $\mathcal{Y}_{k,+\ell,0}$ | \rightarrow | $\mathcal{Y}_{k,+\ell,+1}$ | \rightarrow | \cdots | \rightarrow | $\mathcal{Y}_{k,+\ell,+k/2}$ |
| | | | | | | \uparrow | | | | | | |
| | | | | \vdots | | $Y_{\ell,+\ell}$ | | \vdots | | | | \vdots |
| $\mathcal{Y}_{k,+1,-k/2}$ | \leftarrow | \cdots | \leftarrow | $\mathcal{Y}_{k,+1,-1}$ | \leftarrow | $\mathcal{Y}_{k,+1,0}$ | \rightarrow | $\mathcal{Y}_{k,+1,+1}$ | \rightarrow | \cdots | \rightarrow | $\mathcal{Y}_{k,+1,+k/2}$ |
| | | | | | | \uparrow | | | | | | |
| | | | | | | $Y_{\ell,+1}$ | | | | | | |
| $\mathcal{Y}_{k,0,-k/2}$ | \leftarrow | \cdots | \leftarrow | $\mathcal{Y}_{k,0,-1}$ | \leftarrow | $\mathcal{Y}_{k,0,0}$ | \rightarrow | $\mathcal{Y}_{k,0,+1}$ | \rightarrow | \cdots | \rightarrow | $\mathcal{Y}_{k,0,+k/2}$ |
| | | | | | | \uparrow | | | | | | |
| | | | | | | $Y_{\ell,0}$ | | | | | | |
| $\mathcal{Y}_{k,-1,-k/2}$ | \leftarrow | \cdots | \leftarrow | $\mathcal{Y}_{k,-1,-1}$ | \leftarrow | $\mathcal{Y}_{k,-1,0}$ | \rightarrow | $\mathcal{Y}_{k,-1,+1}$ | \rightarrow | \cdots | \rightarrow | $\mathcal{Y}_{k,-1,+k/2}$ |
| | | | | | | \uparrow | | | | | | |
| | | | | \vdots | | $Y_{\ell,-1}$ | | \vdots | | | | \vdots |
| $\mathcal{Y}_{k,-\ell,-k/2}$ | \leftarrow | \cdots | \leftarrow | $\mathcal{Y}_{k,-\ell,-1}$ | \leftarrow | $\mathcal{Y}_{k,-\ell,0}$ | \rightarrow | $\mathcal{Y}_{k,-\ell,+1}$ | \rightarrow | \cdots | \rightarrow | $\mathcal{Y}_{k,-\ell,+k/2}$ |
| | | | | | | \uparrow | | | | | | |
| | | | | | | $Y_{\ell,-\ell}$ | | | | | | |

Convention 3.4.3. For the remainder of this paper we will assume that all coefficients are chosen in complex-conjugate pairs $c_{k,m,n} = \overline{c_{k,-m,-n}}$ and therefore all modes are real valued.

4. Eigenmodes of spherical spaces S^3/Γ^*

A spherical space is a quotient manifold $M = S^3/G$, with G a finite subgroup of $SO(4)$. An eigenmode of M with eigenvalue $k(k + 2)$ corresponds naturally to a k -eigenmode of S^3 that is G -invariant. The set of all such modes forms a subspace \mathcal{V}_M^k of the vector space \mathcal{V}^k of all k -eigenmodes of S^3 . In the present paper we focus on the case that G is a binary polyhedral group Γ^* , because those spaces hold the greatest interest for cosmology as well as being technically easier.

4.1. Vertical modes of S^3/Γ^* generate all modes of S^3/Γ^*

We will now show that when searching for Γ^* -invariant eigenmodes, we may safely restrict our attention to the vertical ones.

Proposition 4.1.1. Every Γ^* -invariant mode of S^3 may be obtained from vertical Γ^* -invariant modes by applying the twist operators and taking a sum.

Proof. Let F be an arbitrary Γ^* -invariant mode of S^3 (not necessarily vertical). Express F relative to the basis \mathcal{Y}_{kmn} (table 1) as

$$F = \sum_{kmn} c_{kmn} \mathcal{Y}_{kmn} = \sum_{kn} \left(\sum_m c_{kmn} \mathcal{Y}_{kmn} \right) = \sum_{kn} F_{kn}, \quad (21)$$

where $F_{kn} \equiv \sum_m c_{kmn} \mathcal{Y}_{kmn}$ is the component of F that is simultaneously a k -eigenvalue of the Laplace operator Δ_{S^3} and an n -eigenvalue of the twist-measuring operator Z (equation (13)). By assumption each element $\gamma \in \Gamma^*$ preserves F . Because γ commutes with both Δ_{S^3} and Z , it must preserve each F_{kn} individually. (Unlike an arbitrary element of $SO(4)$, the isometry γ commutes with Z because γ takes Clifford parallels to Clifford parallels.) Thus each F_{kn} is Γ^* -invariant.

Because F_{kn} has constant twist, it is easily obtained by applying the twist operator to a vertical function,

$$F_{kn} = \sum_m c_{kmn} \mathcal{Y}_{kmn} = \sum_m c_{kmn} \text{twist}^n \mathcal{Y}_{km0} = \text{twist}^n \left(\sum_m c_{kmn} \mathcal{Y}_{km0} \right), \quad (22)$$

where for negative n , twist^n means $\overline{\text{twist}}^{|n|}$. Because the twist operators twist and $\overline{\text{twist}}$ commute with each γ , each vertical function $\sum_m c_{kmn} \mathcal{Y}_{km0}$ is Γ^* -invariant, thus completing the proof. \square

Like for S^3 , the search for the eigenmodes of S^3/Γ^* reduces to a search for the vertical ones, since each vertical Γ^* -invariant k -eigenmode generates, through the action of the twist operators, a $(k + 1)$ -dimensional vector space of generic Γ^* -invariant k -eigenmodes.

4.2. Modes of S^2/Γ generate all vertical modes of S^3/Γ^*

Section 3.2 showed that the vertical modes of S^3 are the pullbacks of the modes of S . Thus in a direct geometrical sense, the modes of S^2 are the vertical modes of S^3 , and Γ^* -invariance on S^3 corresponds directly to Γ -invariance on S^2 .

Conclusion 4.2.1. The search for Γ^* -invariant eigenmodes of S^3 reduces to the search for Γ -invariant eigenmodes of S^2 .

5. Γ -invariant eigenmodes of S^2

5.1. Multipole vectors

Consider V^ℓ , the vector space of ℓ -eigenmodes. According to Maxwell's multipole vector decomposition of modes [13, 14, 16, 17, 19, 21, 25], we may write each eigenmode $f_\ell \in V^\ell$ as

$$f_\ell(x, y, z) = cr^{2\ell+1} \nabla_{v_\ell} \dots \nabla_{v_2} \nabla_{v_1} \frac{1}{r}, \quad (23)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and the decomposition is well defined up to flipping the signs of the direction vectors $\{v_1, \dots, v_\ell\}$ and the scale factor, two at a time. The ordering of the direction vectors is irrelevant.

Define an equivalence relation on V^ℓ setting two functions f and f' to be equivalent whenever they are nonzero real multiples of each other,

$$f \simeq f' \Leftrightarrow f = cf', c \in \mathbb{R} - \{0\}.$$

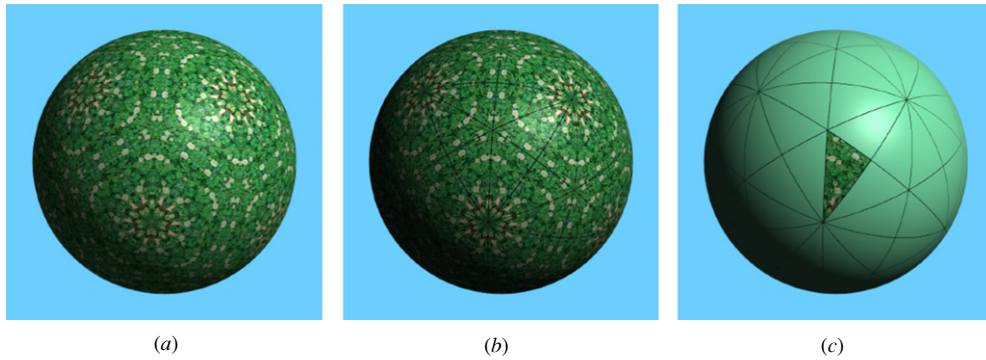


Figure 2. How to construct the *235 orbifold. (a) Begin with an icosahedrally symmetric pattern on the 2-sphere. (b) Locate all lines of mirror symmetry. Each is a great circle, and together they divide the sphere into 120 congruent triangles. (c) Fold the sphere along all mirror lines simultaneously, so that the whole sphere maps 120-to-1 onto a single triangle. The resulting quotient is the *235 orbifold. The Conway notation *235 may be understood as follows: the ‘*’ denotes the mirror-symmetric origin of the triangle’s sides, while the 2, the 3 and the 5 denote the fact that 2, 3 and 5 mirror lines met at each corner, respectively.

All the elements of each equivalence class $[f]$ share the same decomposition (23) up to the choice of signs for the direction vectors $\{v_1, \dots, v_\ell\}$ and the leading constant c . Therefore each equivalence class $[f]$ is uniquely represented by a set of directions $\{d_1, \dots, d_\ell\}$, where each direction d_i represents a line $\pm v_i$, with no concern for the sign. The set of all possible directions forms a real projective plane $\mathbb{R}P^2 = S^2/\pm Id$.

5.2. Invariant sets of directions

A class $[f]$ of modes is Γ -invariant iff the associated set $\{d_1, \dots, d_\ell\}$ is Γ -invariant. Note that although each symmetry $\gamma \in \Gamma$ is nominally a map $\gamma : S^2 \rightarrow S^2$, its action on $\mathbb{R}P^2$ is well defined. To understand the possible classes $[f]$ of Γ -invariant modes, we need to understand the possible Γ -invariant sets $\{d_1, \dots, d_\ell\}$ of directions.

6. Eigenmodes of the Poincaré dodecahedral space S^3/I^*

Let us now further restrict our attention to the PDS, because of the interest it holds in cosmology as well as its greater technical ease. In other words, let Γ be the icosahedral group I comprising the 60 orientation-preserving symmetries of a regular icosahedron. We will consider sets of directions $\{d_1, \dots, d_\ell\}$ that are invariant under I . Because each direction d_i is automatically invariant under the antipodal map, the set $\{d_1, \dots, d_\ell\}$ will be invariant under the full group I_h of 120 symmetries of a regular icosahedron, reflections included.

6.1. The orbifold

The quotient S^2/I_h is an orbifold consisting of a spherical triangle with mirror boundaries and corner reflectors with angles $\pi/2, \pi/3$ and $\pi/5$ (see figure 2). In Conway’s notation this orbifold is denoted *235.

- Each point in the *interior* of the triangle lifts to an invariant set of 120 points on S^2 , which in turn defines an invariant set of 60 directions.

- However, each point on a *mirror boundary* lifts to only 60 points on S^2 , defining only 30 directions. For this reason it is convenient to think of a point on the mirror boundary as a ‘half point’.
- A point at the *corner reflector* of angle $\pi/2$ lifts to 30 points on S^2 or 15 directions, so it is convenient to think of it as a ‘quarter point’.
- Similarly, the points at the corner reflectors of angle $\pi/3$ and $\pi/5$ may be considered a $1/6$ point and a $1/10$ point, respectively.

In all cases, a $\frac{1}{F}$ fractional ($F = 1, 2, 4, 6$ or 10) point of S^2/I_h represents an invariant set of $\ell = \frac{60}{F}$ directions. Another way to think about it is that a half point on the mirror boundary lifts to 120 half points on S^2 , and then each pair of identically positioned half points combines to form a single full point, and similarly for the other fractional points.

Definition 6.1.1. *Let*

- $C_{\frac{1}{10}}$ denote the number of $\frac{1}{10}$ points at the vertex of angle $\pi/5$,
- $C_{\frac{1}{6}}$ denote the number of $\frac{1}{6}$ points at the vertex of angle $\pi/3$,
- $C_{\frac{1}{4}}$ denote the number of $\frac{1}{4}$ points at the vertex of angle $\pi/2$,
- $C_{\frac{1}{2}}$ denote the number of half points on the triangle’s perimeter, and
- C_1 denote the number of whole points in the triangle’s interior.

The preceding discussion has shown that

Proposition 6.1.2. *Each I -invariant equivalence class $[f]$ of modes of S^2 corresponds to a unique choice of ℓI -invariant multipole vectors. The degree of a representative mode f is*

$$\ell = 6C_{\frac{1}{10}} + 10C_{\frac{1}{6}} + 15C_{\frac{1}{4}} + 30C_{\frac{1}{2}} + 60C_1. \tag{24}$$

Some care is required here: knowing that an equivalence class $[f]$ of modes is I -invariant does not immediately imply that each representative f of that class is I -invariant. It is *a priori* possible that some symmetry $\gamma \in I$ could send f to $-f$. The following proposition shows that this does not happen.

Proposition 6.1.3. *If an equivalence class $[f]$ of modes of S^2 is invariant under the icosahedral group I , then each representative f is also invariant under I .*

Proof. Let $\{d_1, \dots, d_\ell\}$ be the set of I -invariant directions defining the class $[f]$ (section 5.2), and let $\gamma \in I$ be a symmetry of the icosahedron. By the assumed I -invariance of $[f]$, we know that γ sends each d_i to $\pm d_j$ (for some j). To prove that f itself is invariant, it suffices to prove that γ sends d_i to $-d_j$ (rather than to $+d_j$) for an even number of d_i .

First, consider the case that a given d_i lies in the ‘interior’ of the $*235$ -orbifold (figure 2(c)). This implies that 59 other d_i (for different values of i) lie in the interiors of other copies of the fundamental triangle (figure 2(b)), arranged symmetrically. Each right-handed copy of the fundamental triangle lies antipodally opposite a left-handed copy (figure 2(b)). If we make the convention to orient each of the 60 d_i in question so that it points toward a right-handed copy of the triangle and away from a left-handed copy, then every $\gamma \in I$ will preserve those d_i exactly, always sending a d_i to a $+d_j$, never to a $-d_j$.

Next, consider the case that some d_i lies on the perimeter (the mirror boundary) of the $*235$ -orbifold’s fundamental triangle. In this case it has only 30 translates under the group (including itself). The icosahedral group I consists entirely of rotations, each about some vertex of the tiling (figure 2(b)). Let γ be some such rotation. In the generic case that none of the 30 d_i lies exactly 90° from the rotation axis of γ , we may orient all 30 d_i to point

toward the ‘northern hemisphere’ (relative to γ ’s rotation axis) and away from the ‘southern hemisphere’. In this case γ sends each d_i to a $+d_j$, never to a $-d_j$. In the non-generic case that some of the d_i lie exactly on the ‘equator’ relative to γ ’s rotation axis, consider the three sub-cases that the rotation γ has order 2, 3 or 5. When γ is a rotation of order 3 or 5, easy ad hoc conventions serve to orient the equatorial d_i so that γ respects their orientations. When γ is a rotation of order 2, it perforce takes each d_i to $-d_i$, but there are exactly two such d_i , so the net effect is still that γ maps the mode f to $+f$, not $-f$.

Finally, consider the case that some d_i lies isolated at one of the fundamental triangle’s vertices. According to whether the vertex is a corner reflector of order 2, 3 or 5, d_i will have 15, 10 or 6 translates (including itself), respectively. Imitating the method of the preceding paragraph, we consider a rotation $\gamma \in I$, and wherever possible orient the d_i to point toward the northern hemisphere and away from the southern hemisphere, thus ensuring that γ permutes such d_i respecting orientation. It remains to consider only the d_i that lie on the equator relative to γ ’s rotation axis. When γ has order 3 or 5, its equator contains corner reflectors of order 2 only, and an ad hoc convention serves to orient them consistently. When γ has order 2, it maps each equatorial d_i to $-d_i$, but the equator contains exactly four corner reflectors of order 2, four corner reflectors of order 3 and four corner reflectors of order 5, so in each sub-case the equator contains exactly two of the directions d_i (from among the complete set of 15, 10 or 6 directions under consideration), and because exactly two directions get flipped, we conclude that γ maps the mode f to $+f$, not $-f$. \square

Corollary 6.1.4. *Any value of ℓ not expressible in the form (24), for example $\ell = 14$, cannot be the degree of an eigenmode of S^2/I .*

Corollary 6.1.5. *The nontrivial I -invariant mode of S^2 of least degree has degree $l = 6$.*

6.2. Dimension of the space of modes

Proposition 6.2.1. *The I -invariant mode of degree $l = 6$ is unique up to a constant multiple. Thus $\dim(V^6) = 1$.*

Proof. To construct this mode, take the *235 orbifold and place a single $1/10$ point at the corner reflector of angle $\pi/5$. This $1/10$ point lifts to 12 points of S^2 which in turn define six directions. According to Maxwell’s formula (23), those six directions define an I -invariant class of modes $[f]$ of degree 6. By proposition 6.1.3, each representative f of $[f]$ is I -invariant. Assuming a fixed realization of the icosahedral group I , the six directions are well defined—they align with the vertices of an icosahedron or the face centers of a dodecahedron. Therefore the class $[f]$ is also well defined, and the only degree of freedom for the mode f is the scale factor inherent in the equivalence class $[f]$. Thus V^6 is of dimension 1. \square

The method of the preceding proposition lets us construct I -invariant modes of degree 10 (place a $1/6$ point at the corner reflector of angle $\pi/3$) and degree 15 (place a $1/4$ point at the corner reflector of angle $\pi/2$), while proving that I -invariant modes of most other low degrees cannot exist. V^{10} and V^{15} have dimension 1.

The case of degree 30, realized by placing a half point on the *235 orbifold’s mirror boundary, is more interesting because we have an extra degree of freedom corresponding to where we choose to place the half point. Allowing for the scale factor inherent in the equivalence class $[f]$ gives a total of two real degrees of freedom: V^{30} has dimension 2.

The case of degree 60, corresponding to one full point in the *235 orbifold, is more interesting still, because now we have a choice as to how we realize that one full point:

- *Case 1.* We may place a single full point anywhere in the orbifold.
- *Case 2.* We may place two half points on the orbifold’s mirror boundary. In the special case that the two half points coincide, we get a single full point as in case 1.
- *Case 3.* We may place any combination of fractional points at the orbifold’s corner reflectors, just so the fractions sum to one. However it turns out that the only ways to do this are to place a full point at a single corner (for example realized as ten $1/10$ points at the corner of angle $\pi/5$) or to place a half point at each of two corners (for example realized as five $1/10$ points at the corner of angle $\pi/5$ plus three $1/6$ points at the corner of angle $\pi/3$). The full point corresponds to case 1 while the two half points correspond to case 2, so nothing new arises here and we will henceforth ignore this case 3.
- *Case 4.* We may place a half point on the mirror boundary and a half-point’s worth of fractional points at the corner reflectors, but as in case 3 nothing new arises here so we may ignore this possibility.

Proposition 6.2.2. *The I -invariant classes $[f]$ of modes of S^2 of degree $l = 60$ are parameterized by a real projective plane.*

Proof. Each class $[f]$ of degree 60 corresponds to 60 directions $\{d_1, \dots, d_{60}\}$ that are invariant under the icosahedral group I , which in turn correspond either to a single point in the *235 orbifold (case 1) or to a pair of half points on the mirror boundary (case 2).

The possible locations for a whole point are obviously parameterized by the points of the orbifold itself, which is topologically a disk.

The possible locations for a pair of points on the orbifold’s mirror boundary are parameterized by a Möbius strip. To see why, first note that the mirror boundary is topologically a circle S^1 . Parameterize this circle in some arbitrary but fixed way, with the parameter angle defined modulo 2π , and then for any pair of points define

- θ = the position of the two points’ ‘center of mass’ ($\theta \in S^1 = \mathbb{R}/2\pi$).
- ϕ = the separation between the two points ($\phi \in [0, \pi]$).

At first glance this gives a cylinder parameterized by (θ, ϕ) . But (θ, π) and $(\theta + \pi, \pi)$ define the same pair of points, so we must identify opposite points on the cylinder’s upper boundary circle $(\theta, \pi) \sim (\theta + \pi, \pi)$, which transforms the cylinder into a Möbius strip. The cylinder’s lower boundary circle $(\theta, 0)$ becomes the Möbius strip’s edge.

The Möbius strip’s edge, parameterized by $(\theta, 0)$, corresponds to the case that the two half points fuse together to form a single whole point on the triangle’s perimeter. This corresponds exactly to the boundary of the disk in the whole point parameter space. In other words, the total parameter space is the union of a disk and a Möbius strip glued together along their boundary circles, which yields a real projective plane. \square

It is no surprise that the parameter space is a real projective plane. The space of Γ -invariant harmonic functions f on S^2 of any fixed degree ℓ is a vector space of some finite dimension n . When we pass from functions f to equivalence classes $[f]$ we identify each line through the origin to a single point, giving in all cases a real projective space $\mathbb{R}P^{n-1}$. In the case just considered, with degree $\ell = 60$, we found the projective space to be $\mathbb{R}P^2$ meaning the total function space, including the scale factor, is \mathbb{R}^3 .

To construct a generic I -invariant mode, we may place any combination of whole points (anywhere in the orbifold), half points (on the orbifold’s mirror boundary) and other fractional points (isolated at the orbifold’s corner reflectors). Each whole point contributes two degrees of freedom to the space of modes (corresponding to the point’s location in the two-dimensional triangle), each half point contributes one degree of freedom (corresponding to its location

along the triangle’s one-dimensional perimeter), and each isolated fractional point contributes nothing. The overall scaling factor contributes one more degree of freedom for any nontrivial mode. In summary,

Proposition 6.2.3. *The dimension of the space of I-invariant ℓ -eigenmodes of S^2 is given by*

$$\dim(V^\ell) = 1 + C_{\frac{1}{2}} + 2C_1. \tag{25}$$

Note that no matter how many half points may or may not combine into whole points, the half and whole points together contribute $C_{\frac{1}{2}} + 2C_1$ degrees of freedom.

6.3. Improved dimension formula

The dimension formula (25) is nice, but we would much rather have a formula in terms of ℓ , to save us the trouble of manually decomposing ℓ into a linear combination of the C_i . Here is the improved formula.

Proposition 6.3.1. *The dimension of the space of I-invariant ℓ -eigenmodes of S^2 is given by*

$$\dim(V^\ell) = 1 + \left\lfloor \frac{\ell}{2} \right\rfloor + \left\lfloor \frac{\ell}{3} \right\rfloor + \left\lfloor \frac{\ell}{5} \right\rfloor - \ell. \tag{26}$$

Proof. Recall that

$$\ell = 6C_{\frac{1}{10}} + 10C_{\frac{1}{6}} + 15C_{\frac{1}{4}} + 30C_{\frac{1}{2}} + 60C_1. \tag{27}$$

and consider how the C_i depend on ℓ .

First consider $C_{\frac{1}{10}}$, the number of $\frac{1}{10}$ points. Taking equation (27) modulo 5 we get

$$\ell \equiv C_{\frac{1}{10}} \pmod{5}.$$

But the number of isolated $\frac{1}{10}$ points may only be 0, 1, 2, 3 or 4, because if we had 5 or more $\frac{1}{10}$ points they would combine to form half points and acquire an additional degree of freedom. So the number of isolated $\frac{1}{10}$ points must be $C_{\frac{1}{10}} = \ell - 5\lfloor \frac{\ell}{5} \rfloor$. The same argument, repeated mod 3 and mod 2, gives $C_{\frac{1}{6}} = \ell - 3\lfloor \frac{\ell}{3} \rfloor$ and $C_{\frac{1}{4}} = \ell - 2\lfloor \frac{\ell}{2} \rfloor$, respectively.

Rearranging equation (27) now gives

$$\begin{aligned} C_{\frac{1}{2}} + 2C_1 &= \frac{1}{30}[\ell - 6C_{\frac{1}{10}} - 10C_{\frac{1}{6}} - 15C_{\frac{1}{4}}] \\ &= \frac{1}{30} \left[\ell - 6 \left(\ell - 5 \left\lfloor \frac{\ell}{5} \right\rfloor \right) - 10 \left(\ell - 3 \left\lfloor \frac{\ell}{3} \right\rfloor \right) - 15 \left(\ell - 2 \left\lfloor \frac{\ell}{2} \right\rfloor \right) \right] \\ &= \left\lfloor \frac{\ell}{5} \right\rfloor + \left\lfloor \frac{\ell}{3} \right\rfloor + \left\lfloor \frac{\ell}{2} \right\rfloor - \ell. \end{aligned} \tag{28}$$

Substituting equation (28) into equation (25) gives the final result (26) as stated above. \square

This agrees with Ikeda’s formula [5], while at the same time providing a concrete construction of the modes and shedding additional light on the formula’s geometrical origins, as degrees of freedom in an orbifold.

7. Conclusion

Returning to the three-dimensional PDS S^3/I^* , the results of the preceding sections may be summarized as follows. Keep in mind that S^3/I^* admits k -modes for even k only; odd k -modes cannot exist because I^* contains the antipodal map.

Theorem 7.1. *To construct the modes of the PDS S^3/I^* ,*

- *Each mode of S^3/I^* corresponds to an I^* -invariant mode of S^3 (elementary).*
- *Each I^* -invariant mode of S^3 is a sum of twists of I^* -invariant vertical modes of S^3 (proposition 4.1.1).*
- *Each I^* -invariant vertical k -mode of S^3 is the pull-back, under the Hopf map, of an I -invariant ℓ -mode of S^2 , with $k = 2\ell$ (proposition 3.2.1).*
- *The I -invariant ℓ -modes of S^2 are parameterized by $\ell/60$ points on the $*235$ -orbifold, possibly including fractional points (section 6.1).*

Theorem 7.2. *The space of k -modes of the PDS S^3/I^* has the dimension*

$$(k + 1) \left(1 + \left\lfloor \frac{k/2}{2} \right\rfloor + \left\lfloor \frac{k/2}{3} \right\rfloor + \left\lfloor \frac{k/2}{5} \right\rfloor - \frac{k}{2} \right). \quad (29)$$

Proof. The space of I -invariant $k/2$ -modes of the 2-sphere has the dimension $1 + \left\lfloor \frac{k/2}{2} \right\rfloor + \left\lfloor \frac{k/2}{3} \right\rfloor + \left\lfloor \frac{k/2}{5} \right\rfloor - \frac{k}{2}$ (proposition 6.3.1) and thus the space of vertical I^* -invariant k -modes of the 3-sphere has this same dimension (theorem 7.1). The twist operators then take each vertical mode to a $(k+1)$ -dimensional space of generic I -invariant modes (table 1 and proposition 4.1.1), completing the proof. \square

References

- [1] Aurich R 2008 A spatial correlation analysis for a toroidal universe *Preprint* [arXiv:0803.2130v1](http://arxiv.org/abs/0803.2130v1)
- [2] Aurich R, Lustig S, Steiner F and Then H 2006 CMB alignment in multi-connected universes <http://arxiv.org/abs/astro-ph/0612308>
- [3] Bielewicz P and Riazuelo A 2008 The study of topology of the universe using multipole vectors *Preprint* [arXiv:0804.2437v1](http://arxiv.org/abs/0804.2437v1)
- [4] Lachi ze-Rey M and Luminet J-P 1994 Cosmic topology *Phys. Rep.* **254** 3 <http://arxiv.org/abs/gr-qc/9605010v2>
- [5] Ikeda A 1995 On the spectrum of homogeneous spherical space forms *Kodai Math. J.* **18** 57–67
- [6] Lachi ze-Rey M and Caillerie S 2005 Laplacian eigenmodes for spherical spaces *Class. Quantum Grav.* **22** 695–708 <http://arxiv.org/abs/astro-ph/0501419>
- [7] Caillerie S, Lachize-Rey M, Luminet J-P, Lehoucq R, Riazuelo A and Weeks J 2007 A new analysis of Poincar  dodecahedral space model *Preprint* [arXiv:0705.0217](http://arxiv.org/abs/0705.0217)
- [8] Riazuelo A, Caillerie S, Lachize-Rey M, Lehoucq R and Luminet J-P 2006 Constraining cosmic topology with CMB polarization *Preprint* [astro-ph/0601433](http://arxiv.org/abs/astro-ph/0601433)
- [9] Caillerie S, Lachize-Rey M, Luminet J-P, Lehoucq R, Riazuelo A and Weeks J 2007 A new analysis of Poincar  dodecahedral space model *Preprint* [arXiv:0705.0217](http://arxiv.org/abs/0705.0217)
- [10] Gundermann J 2005 Predicting the CMB power spectrum for binary polyhedral spaces <http://arxiv.org/abs/astro-ph/0503014>
- [11] Lachi ze-Rey M 2004 Eigenmodes of dodecahedral space *Class. Quantum Grav.* **21** 2455–64 <http://arxiv.org/abs/gr-qc/0402035>
- [12] Weeks J 2006 Exact polynomial eigenmodes for homogeneous spherical 3-manifolds *Class. Quantum Grav.* **23** 6971–88 <http://arxiv.org/abs/math/0502566v3>
- [13] Maxwell J C 1873 *A Treatise on Electricity and Magnetism* 1st edn (New York: Dover) pp 1954
Maxwell J C 1891 *A Treatise on Electricity and Magnetism* 3rd edn (New York: Dover) pp 1954
- [14] Dennis M R 2004 *J. Phys. A: Math. Gen.* **37** 9487 (*Preprint* [math-ph/0408046](http://arxiv.org/abs/math-ph/0408046))
Dennis M R 2005 *J. Phys. A: Math. Gen.* **38** 1653 <http://arxiv.org/abs/math-ph/0410004>

- [15] Dennis M R and Land K 2007 Probability density of the multipole vectors for a Gaussian cosmic microwave background *Mon. Not. R. Astron. Soc.* **112** (*Preprint* [arXiv:0704.3657v2](http://arxiv.org/abs/0704.3657v2))
- [16] Copi C J, Huterer D and Starkman G D 2004 *Phys. Rev. D* **70** 043515 <http://arxiv.org/abs/astro-ph/0310511>
- [17] Katz G and Weeks J 2004 *Phys. Rev. D* **70** 063527 <http://arxiv.org/abs/astro-ph/0405631>
- [18] Komatsu E *et al* Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological interpretation http://lambda.gsfc.nasa.gov/product/map/dr3/pub_papers/fiveyear/cosmology/wmap_5yr_cosmo.pdf
- [19] Land K and Magueijo J 2005a *Phys. Rev. Lett.* **95** 071301 <http://arxiv.org/abs/astro-ph/0502237>
- [20] Luminet J-P, Weeks J, Riazuelo A, Lehoucq R and Uzan J-P 2003 Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background *Preprint* [astro-ph/0310253](http://arxiv.org/abs/astro-ph/0310253)
- [21] Lachi ze-Rey M 2004 <http://arxiv.org/abs/astro-ph/0409081>
- [22] Rakic A and Schwarz D 2007 *Phys. Rev. D* **75** 103002 (*Preprint* [astro-ph/0703266](http://arxiv.org/abs/astro-ph/0703266))
- [23] Lew B S and Roukema B F 2008 A test of the poincar  dodecahedral space topology hypothesis with the WMAP CMB data *Preprint* [arXiv:0801.1358](http://arxiv.org/abs/0801.1358)
- [24] Roukema B F, Bulinski Z, Szaniewska A and Gaudin N E 2008 The optimal phase of the generalised poincare dodecahedral space hypothesis implied by the spatial cross-correlation function of the WMAP sky maps *Preprint* [arXiv:0801.0006](http://arxiv.org/abs/0801.0006)
- [25] Weeks J R 2004 <http://arxiv.org/abs/astro-ph/0412231>